# Combinatorial Sanitization of Strings

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**CPM 2026 Summer School**

# Outline

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**The (informal) goal of String Sanitization**

Conceal **sensitive** patterns in $W$ while maintaining **data utility**.

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**C3** is implied by **C2**.

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**Theorem**

**TFS** *can be solved in* $O(||\mathcal{S}|| + k|W|)$ *time. We have* $|X| = \Omega(k|W|)$.

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**MVR** (Missing Value Replacement) problem

Given strings $U, V \in \Sigma^*$ and a finite set $\mathcal{S} \subset \Sigma^*$, compute a shortest string $X \in \Sigma^*$: $U$ is a prefix of $X$, $V$ is a suffix of $X$, and no $S \in \mathcal{S}$ occurs in $X$.

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$$U = \mathtt{aab} \quad V = \mathtt{aba}$$
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# Suffix tree

The compacted trie of all the suffixes of the string.

$$T = \overset{0\,1\,2\,3\,4\,5\,6}{\texttt{CAGAGA\#}}$$

0 : CAGAGA#
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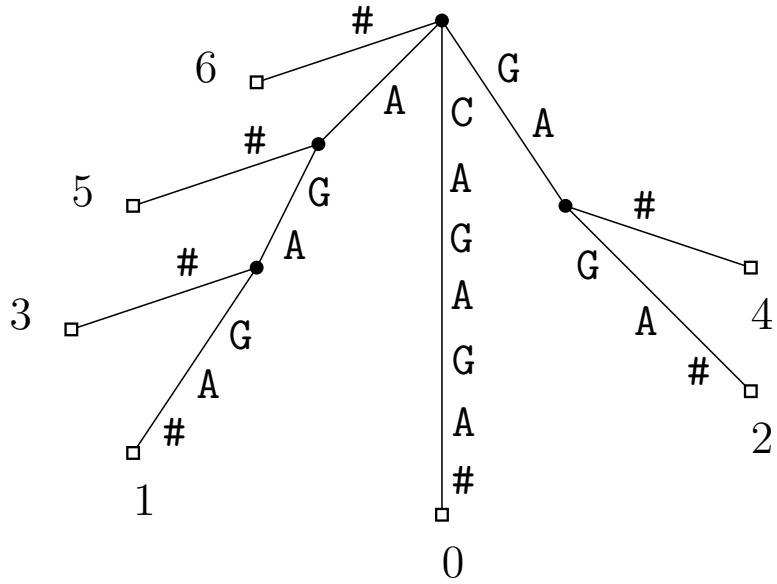
6 : #




We assume that the terminating symbol # is lex-smallest.

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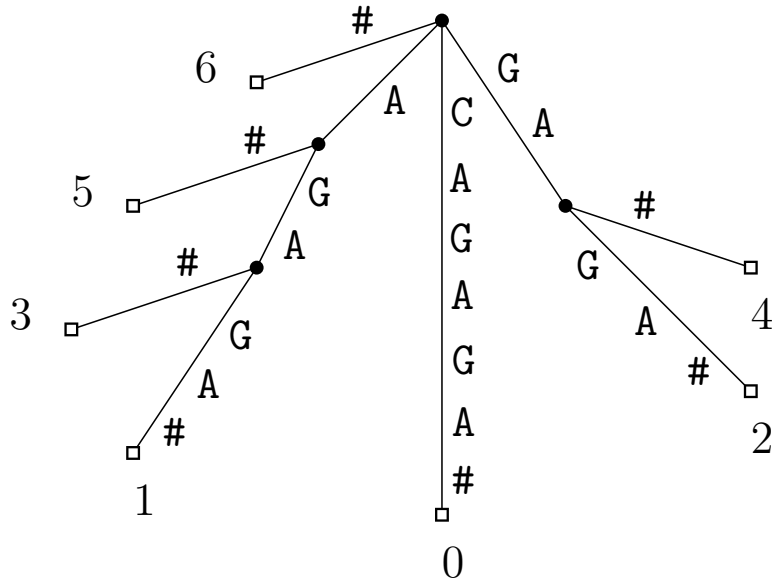


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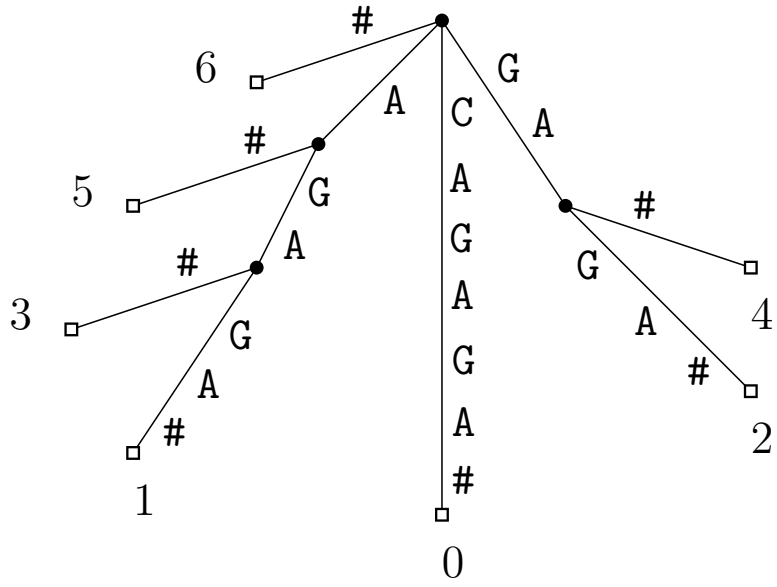


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**Theorem (Weiner, FOCS 1973)**

*The suffix tree of $T$ occupies $O(n)$ space.*

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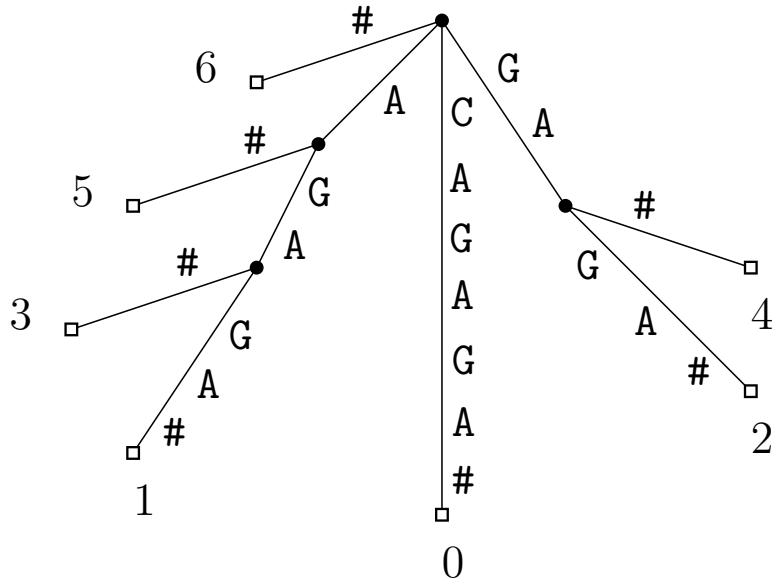


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**Theorem (Farach, FOCS 1997)**

*The suffix tree of $T$ can be constructed in $O(n)$ time.*

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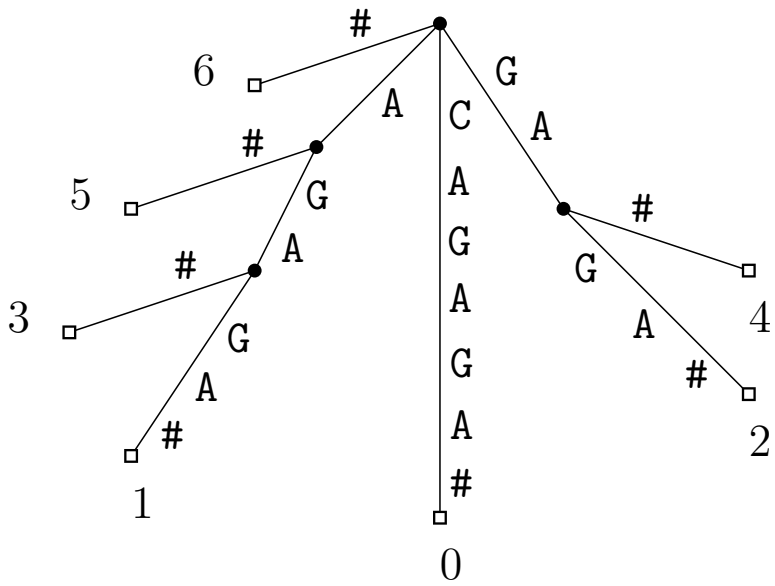

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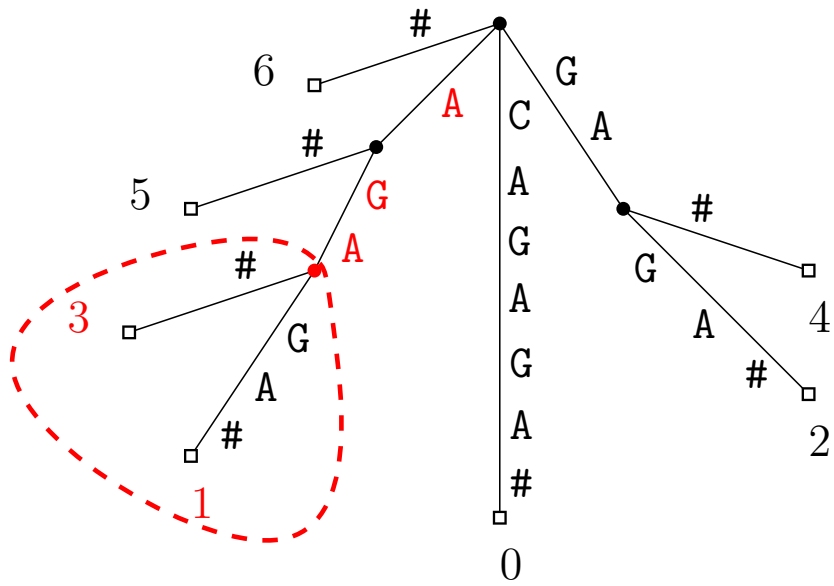

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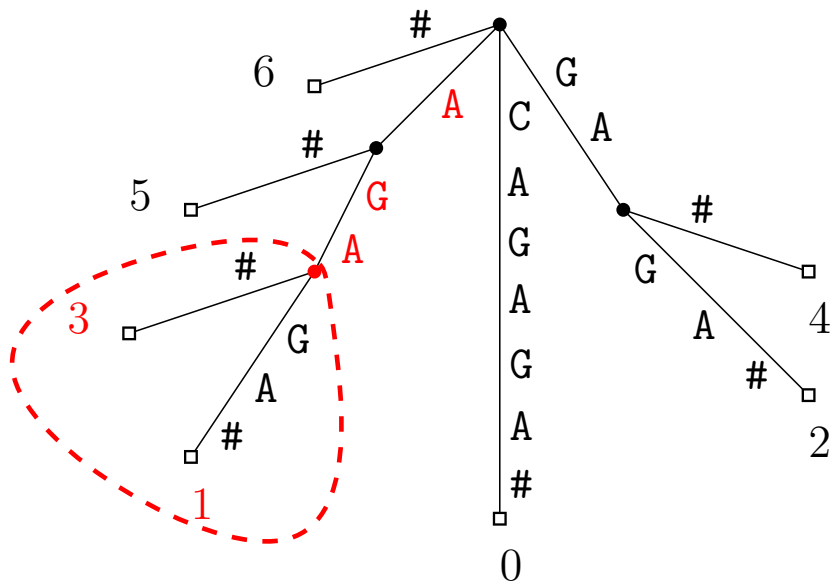


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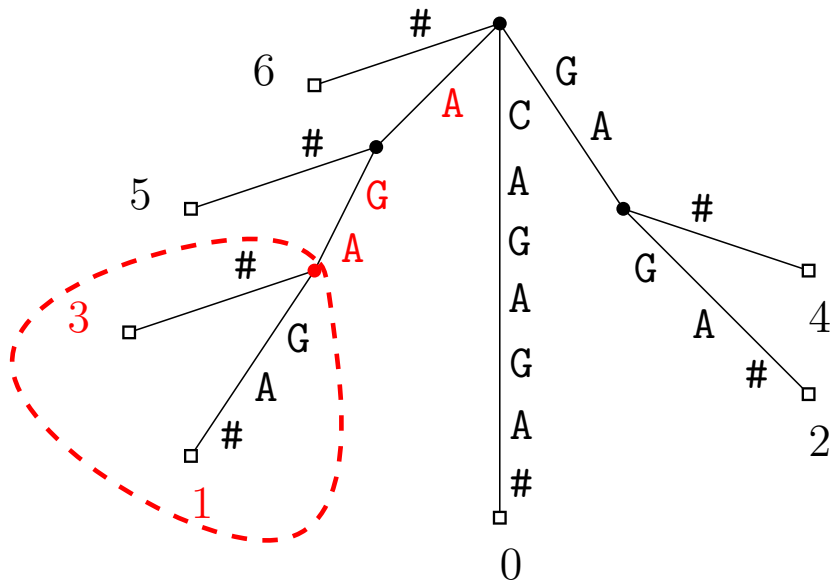


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**Theorem (Weiner, FOCS 1973)**

*The occurrences of pattern $P$ in $T$ can be found in $O(|P| + \text{output})$ time.*

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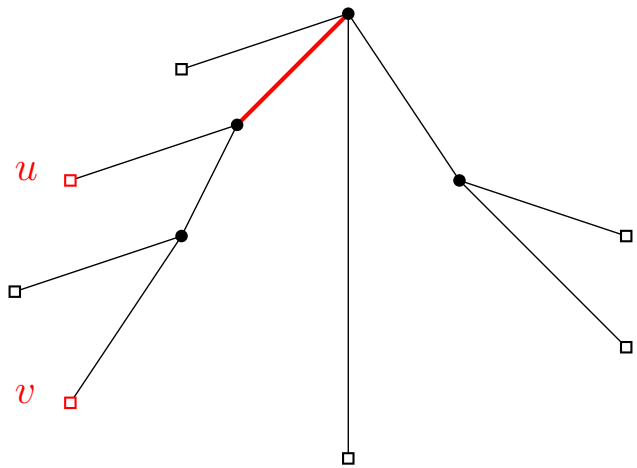

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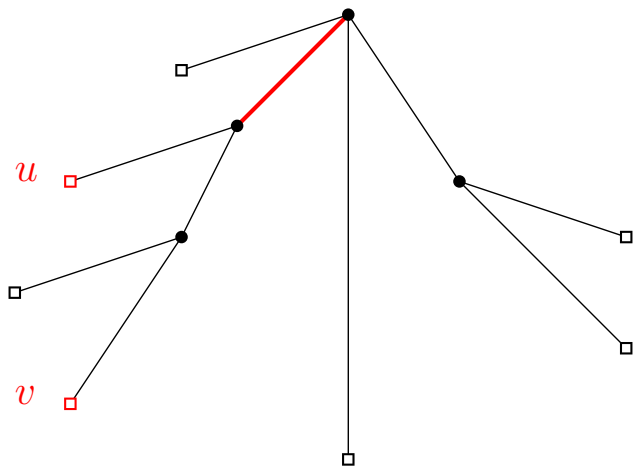


**Theorem (Bender and Farach-Colton, LATIN 2000)**

*Any tree of size $O(N)$ can be preprocessed in $O(N)$ time and space so that the LCA of any two nodes can be computed in $O(1)$ time.*

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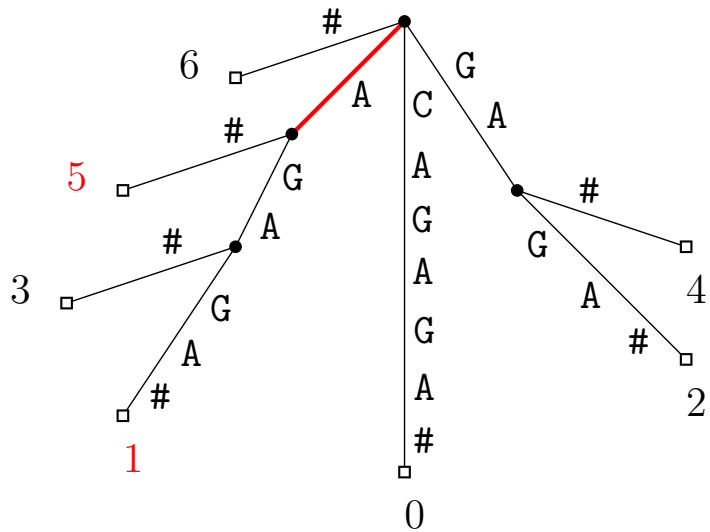

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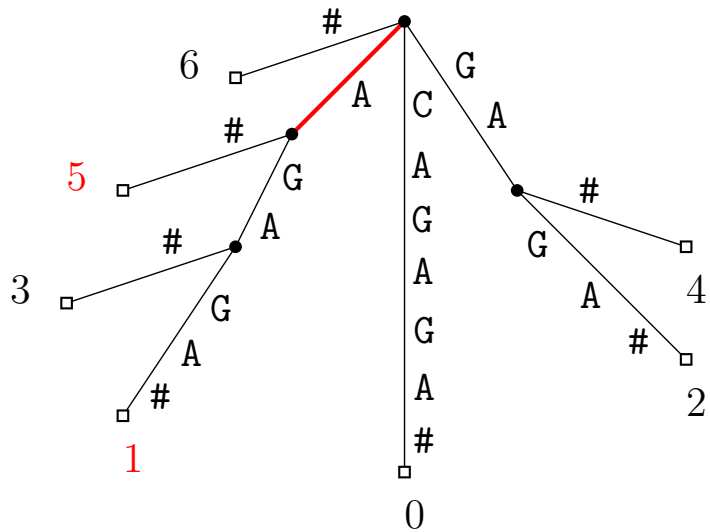


**Theorem (Landau and Vishkin, TCS 1986)**

*LCP queries in $T$ can be answered in $O(1)$ time after $O(n)$ time preprocessing.*

# Solving TFS: Intuition

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  - The check is implemented in $O(1)$ time using LCP queries

# Algorithm Pseudocode

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We find the occurrences of the sensitive patterns using the suffix tree.

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# Algorithm Pseudocode

## Preprocessing: $O(|W| + ||\mathcal{S}||)$ time

We find the occurrences of the sensitive patterns using the suffix tree.

1: Append the first non-sens. pattern to $X$
2: Examine two consecutive patterns

# Algorithm Pseudocode

## Preprocessing: $O(|W| + ||\mathcal{S}||)$ time

We find the occurrences of the sensitive patterns using the suffix tree.

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3: **if** both are non-sens. **then** append the last letter of the rightmost to $X$

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6: **if** the leftmost is sens. and the rightmost is not **then**

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We find the occurrences of the sensitive patterns using the suffix tree.

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8:  Append the last letter of the latter to $X$
9:  **else**
10:  Append # and the current non-sens. pattern to $X$

# Full Example

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- Sensitive ($S$): $\mathcal{S} = \{\texttt{baaa}, \texttt{aaaa}, \texttt{bbaa}\}$

# Full Example

**Given:**

- $W = \texttt{aabaaaababbbaab}$, $k = 4$
- Sensitive ($S$): $\mathcal{S} = \{\texttt{baaa}, \texttt{aaaa}, \texttt{bbaa}\}$
- Non-sensitive ($N$): All other length-4 substrings

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**Initialization:**

# Full Example

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**Initialization:**

- The first $N$ pattern is $P_1 = \texttt{aaba}$

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- The first $N$ pattern is $P_1 = \texttt{aaba}$
- Append $P_1$ to $X$

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**Given:**

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- Sensitive ($S$): $\mathcal{S} = \{\texttt{baaa}, \texttt{aaaa}, \texttt{bbaa}\}$
- Non-sensitive ($N$): All other length-4 substrings

**Initialization:**

- The first $N$ pattern is $P_1 = \texttt{aaba}$
- Append $P_1$ to $X$

**Initial Status**

$X = \texttt{aaba}$

# Full Example

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**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

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# Full Example

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**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

- **Condition**: Rightmost is $S$

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**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

- **Condition**: Rightmost is $S$
- **Action**: Mark $P_3$ as last $S$ pattern and advance

# Full Example

**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

- **Condition**: Both are $N$
- **Action**: Append last letter of rightmost (a) to $X$
- $X =$ aabaa

**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

- **Condition**: Rightmost is $S$
- **Action**: Mark $P_3$ as last $S$ pattern and advance
- $X =$ aabaa (unchanged)

# Full Example

**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

- **Condition**: Both are $N$
- **Action**: Append last letter of rightmost (a) to $X$
- $X =$ aabaa

**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

- **Condition**: Rightmost is $S$
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- $X =$ aabaa (unchanged)

**Step 3: Examine $P_3$ (baaa, $S$) and $P_4$ (aaaa, $S$)**

# Full Example

**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

- **Condition**: Both are $N$
- **Action**: Append last letter of rightmost (a) to $X$
- $X = \texttt{aabaa}$

**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

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**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

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- **Action**: Append last letter of rightmost (a) to $X$
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- $X =$ aabaa (unchanged)

**Step 3: Examine $P_3$ (baaa, $S$) and $P_4$ (aaaa, $S$)**

- **Condition**: Both are $S$
- **Action**: Advance

# Full Example

**Step 1: Examine $P_1$ (aaba, $N$) and $P_2$ (abaa, $N$)**

- **Condition**: Both are $N$
- **Action**: Append last letter of rightmost (a) to $X$
- $X =$ aabaa

**Step 2: Examine $P_2$ (abaa, $N$) and $P_3$ (baaa, $S$)**

- **Condition**: Rightmost is $S$
- **Action**: Mark $P_3$ as last $S$ pattern and advance
- $X =$ aabaa (unchanged)

**Step 3: Examine $P_3$ (baaa, $S$) and $P_4$ (aaaa, $S$)**

- **Condition**: Both are $S$
- **Action**: Advance
- $X =$ aabaa (unchanged)

# Full Example

# Full Example

**Step 4: Examine $P_4$ (aaaa, $S$) and $P_5$ (aaab, $N$)**

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- **Condition**: Leftmost is $S$, rightmost is not

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**Step 4: Examine $P_4$ (aaaa, $S$) and $P_5$ (aaab, $N$)**

- **Condition**: Leftmost is $S$, rightmost is not
- **Check**: Do the last marked $S$ ($P_3$: baaa) and current $N$ ($P_5$: aaab) have LCP $(k-1)$?

# Full Example

**Step 4: Examine $P_4$ (aaaa, $S$) and $P_5$ (aaab, $N$)**

- **Condition**: Leftmost is $S$, rightmost is not
- **Check**: Do the last marked $S$ ($P_3$: baaa) and current $N$ ($P_5$: aaab) have LCP $(k-1)$?
  - No

# Full Example

**Step 4: Examine $P_4$ (aaaa, $S$) and $P_5$ (aaab, $N$)**

- **Condition**: Leftmost is $S$, rightmost is not
- **Check**: Do the last marked $S$ ($P_3$: baaa) and current $N$ ($P_5$: aaab) have LCP $(k-1)$?
  - No
- **Action**: Append # and $P_5$ to $X$

# Full Example

**Step 4: Examine $P_4$ (aaaa, $S$) and $P_5$ (aaab, $N$)**

- **Condition**: Leftmost is $S$, rightmost is not
- **Check**: Do the last marked $S$ ($P_3$: baaa) and current $N$ ($P_5$: aaab) have LCP $(k-1)$?
  - No
- **Action**: Append # and $P_5$ to $X$

**Current Status**

$X =$ aabaa#aaab

# Full Example

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**Steps 5 to 9: Sequence of $N$ patterns**

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- $P_6$ (aaba), $P_7$ (abab), $P_8$ (babb), $P_9$ (abbb), $P_{10}$ (bbba)

# Full Example

**Steps 5 to 9: Sequence of $N$ patterns**

- $P_6$ (aaba), $P_7$ (abab), $P_8$ (babb), $P_9$ (abbb), $P_{10}$ (bbba)
- **Condition**: In all pairs, both are $N$

# Full Example

**Steps 5 to 9: Sequence of $N$ patterns**

- $P_6$ (aaba), $P_7$ (abab), $P_8$ (babb), $P_9$ (abbb), $P_{10}$ (bbba)
- **Condition**: In all pairs, both are $N$
- **Action**: Append the last letter of each to $X$

# Full Example

**Steps 5 to 9: Sequence of $N$ patterns**

- $P_6$ (aaba), $P_7$ (abab), $P_8$ (babb), $P_9$ (abbb), $P_{10}$ (bbba)
- **Condition**: In all pairs, both are $N$
- **Action**: Append the last letter of each to $X$

## Current Status

$X = \texttt{aabaa\#aaaba} \quad (P_6)$
$X = \texttt{aabaa\#aaabab} \quad (P_7)$
$X = \texttt{aabaa\#aaababb} \quad (P_8)$
$X = \texttt{aabaa\#aaababbb} \quad (P_9)$
$X = \texttt{aabaa\#aaababbba} \quad (P_{10})$

# Full Example

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**Step 10: Examine $P_{10}$ (bbba, $N$) and $P_{11}$ (bbaa, $S$)**

- **Condition**: Rightmost is $S$
- **Action**: Mark $P_{11}$ as last $S$ pattern and advance

# Full Example

**Step 10: Examine $P_{10}$ (bbba, $N$) and $P_{11}$ (bbaa, $S$)**

- **Condition**: Rightmost is $S$
- **Action**: Mark $P_{11}$ as last $S$ pattern and advance

**Step 11: Examine $P_{11}$ (bbaa, $S$) and $P_{12}$ (baab, $N$)**

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**Step 10: Examine $P_{10}$ (bbba, $N$) and $P_{11}$ (bbaa, $S$)**

- **Condition**: Rightmost is $S$
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- **Action**: Mark $P_{11}$ as last $S$ pattern and advance

**Step 11: Examine $P_{11}$ (bbaa, $S$) and $P_{12}$ (baab, $N$)**

- **Condition**: Leftmost is $S$, rightmost is not
- **Check**: Do the last marked $S$ ($P_{11}$: bbaa) and current $N$ ($P_{12}$: baab) have an LCP of length $k-1$?

# Full Example

**Step 10: Examine $P_{10}$ (bbba, $N$) and $P_{11}$ (bbaa, $S$)**

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  - No
- **Action**: Append # and $P_{12}$ to $X$

# Full Example

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  - No
- **Action**: Append # and $P_{12}$ to $X$

**Output**

$X =$ aabaa#aaababbba#baab

# Asymptotic Optimality: Lower Bound Construction

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*We have* $|X| = \Omega(k|W|)$ *in the worst case.*

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Can you think of such an instance?

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- Let $W$ be a **de Bruijn sequence** of order $(k-1)$ over $\Sigma$

# Asymptotic Optimality: Lower Bound Construction

**Lemma**

*We have* $|X| = \Omega(k|W|)$ *in the worst case.*

**Question**

Can you think of such an instance?

**Construction:**

- Let $W$ be a **de Bruijn sequence** of order $(k-1)$ over $\Sigma$
- By definition: Every string in $\Sigma^{k-1}$ occurs **exactly once** in $W$

# Asymptotic Optimality: Lower Bound Construction

**Lemma**

*We have* $|X| = \Omega(k|W|)$ *in the worst case.*

**Question**

Can you think of such an instance?

**Construction:**

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As a consequence, the sequence of non-sensitive patterns $\langle P_1, P_3, P_5, \ldots \rangle$ contains roughly $|W|/2$ elements.

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- Therefore, $W[i+1 \ldots i+k-1] \neq W[i+2 \ldots i+k]$

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**Theorem**

**TFS** *can be solved in* $O(||\mathcal{S}|| + k|W|)$ *time. We have* $|X| = \Omega(k|W|)$.

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  - Solve SCS for a collection of two-letter strings.

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6. Construct $Y$ using the output superstring

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**return** $P$

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## Observation

Since $|E|$ is fixed, minimizing the length of the superstring is equivalent to **minimizing the number of trails** $|P|$.

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- Isolated cycles cannot be spliced and require 1 new trail

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## Theorem

**PFS** *can be solved in* $O(||\mathcal{S}|| + |W| + |Y|)$ *time.*

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**Collection $B'$ of 2-letter strings**

$X_1 \implies 2 \cdot 3 \qquad X_2 \implies 1 \cdot 4 \qquad X_3 \implies 3 \cdot 2$

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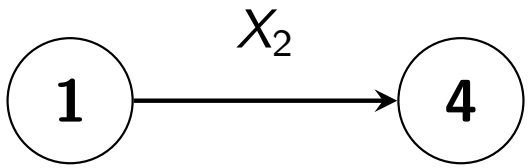


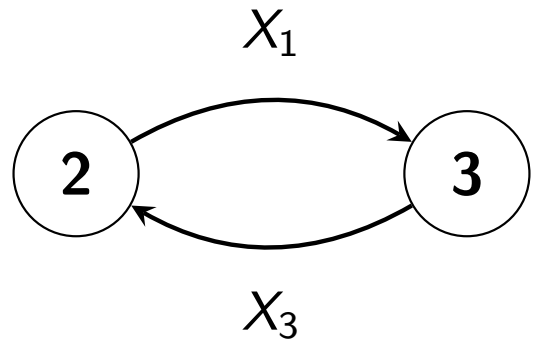

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$P = \{p_1, p_2\}$, where $p_1 = 14$ and $p_2 = 232$.

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**Output**

$Y = \texttt{aaababbba\#aabaab}$

# Reference

G. Bernardini, H. Chen, A. Conte, R. Grossi, G. Loukides, N. Pisanti, S. P. Pissis, G. Rosone, M. Sweering:
Combinatorial Algorithms for String Sanitization.
ACM Trans. Knowl. Discov. Data 15(1): 8:1-8:34 (2021)

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- BFS (from source to sinks) to find a shortest path

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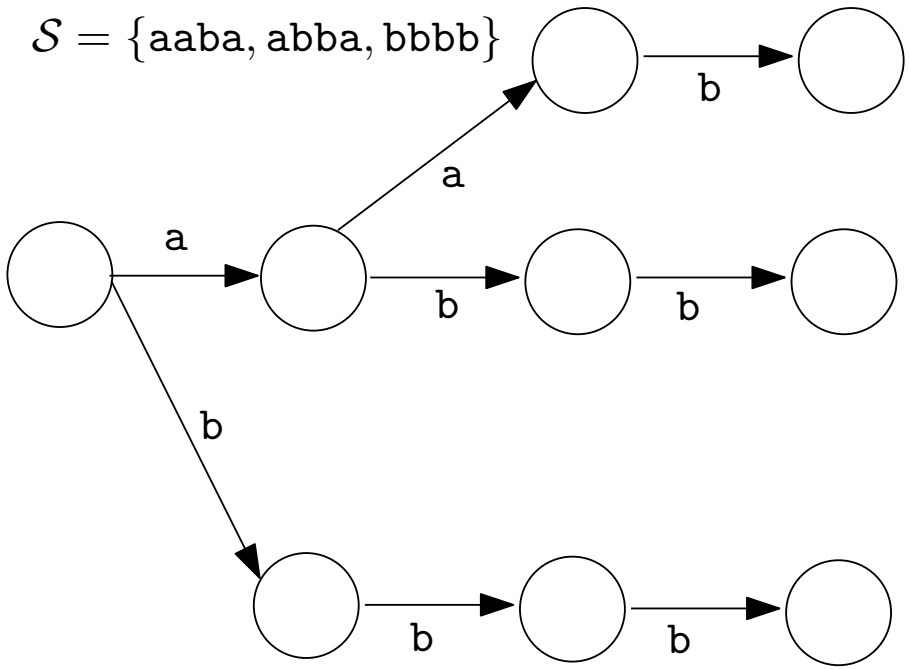

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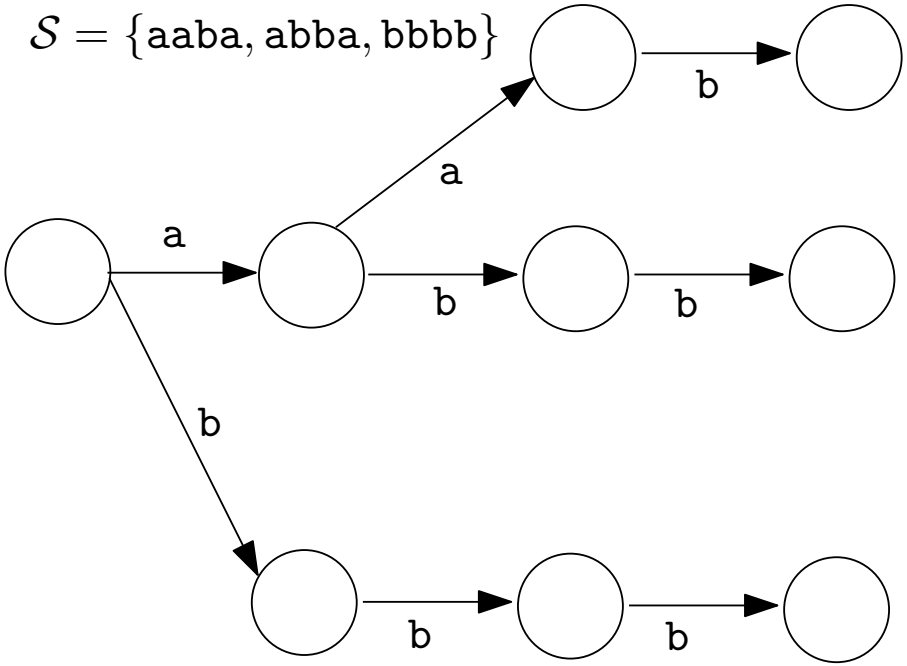


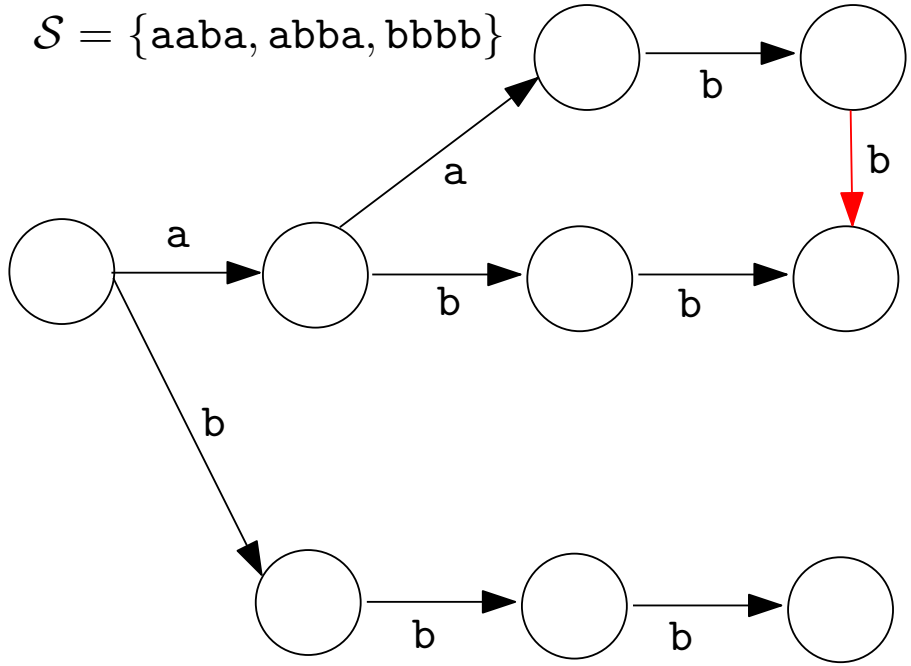

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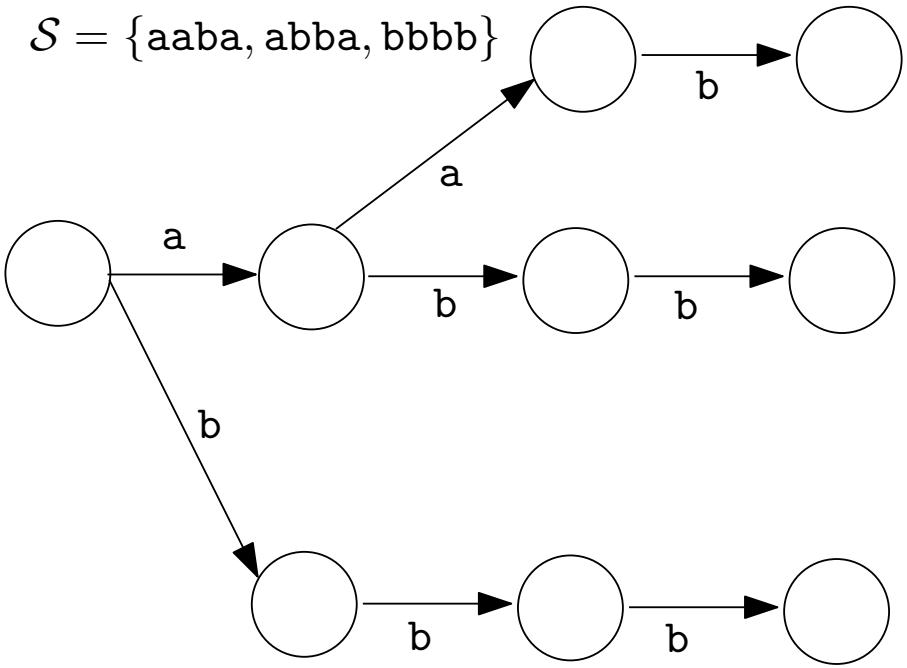


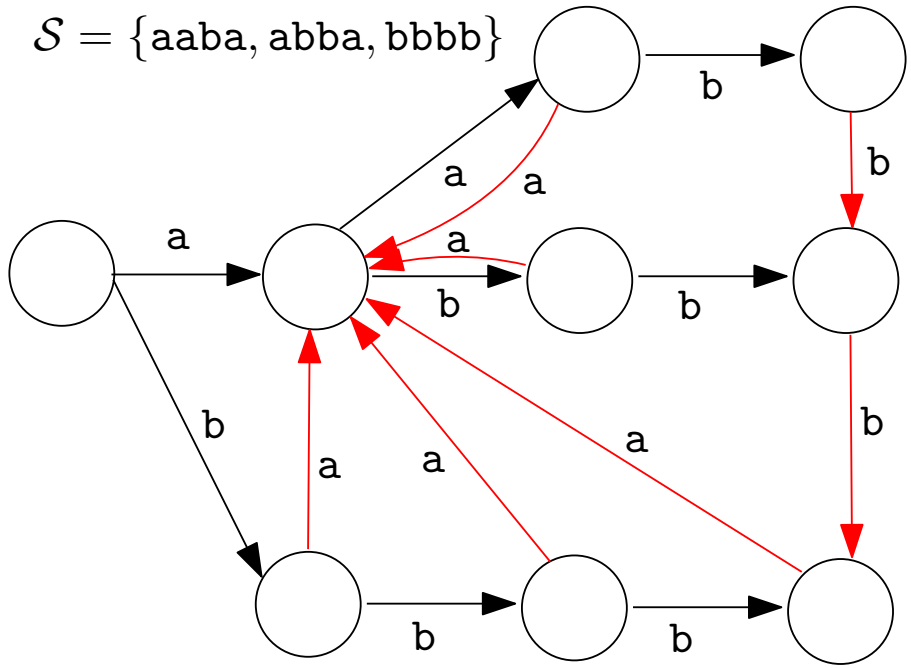


**Lemma**

*$G = (D, E)$ can be constructed in $O(||\mathcal{S}|| \cdot |\Sigma|)$ time.*

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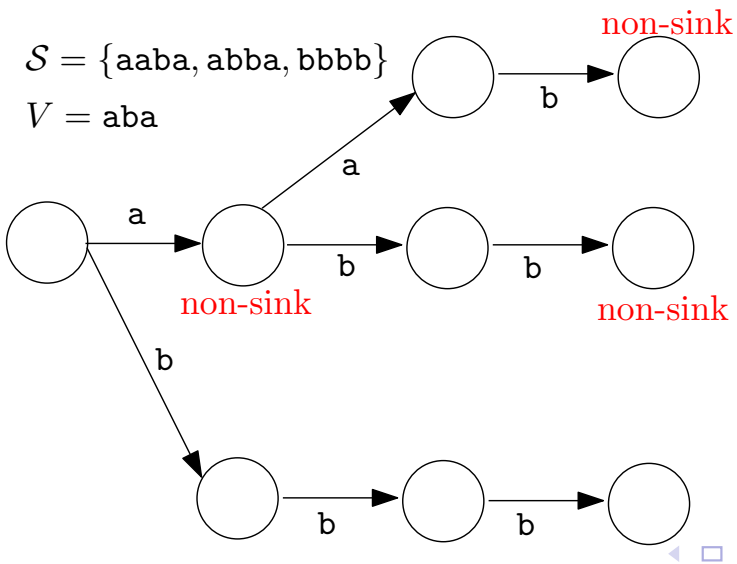

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**Lemma**

*Finding the source and all sinks takes $O(|U| + |V| + ||\mathcal{S}||)$ time.*

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**Theorem**

**MVR** *can be solved in* $O(|U| + |V| + ||\mathcal{S}|| \cdot |\Sigma|)$ *time.*

# Reference

G. Bernardini, C. Liu, G. Loukides, A. Marchetti-Spaccamela, S. P. Pissis, L. Stougie, M. Sweering:
Missing value replacement in strings and applications.
Data Min. Knowl. Discov. 39(2): 12 (2025)

# Outline

# Introduction to $z$-Anonymity

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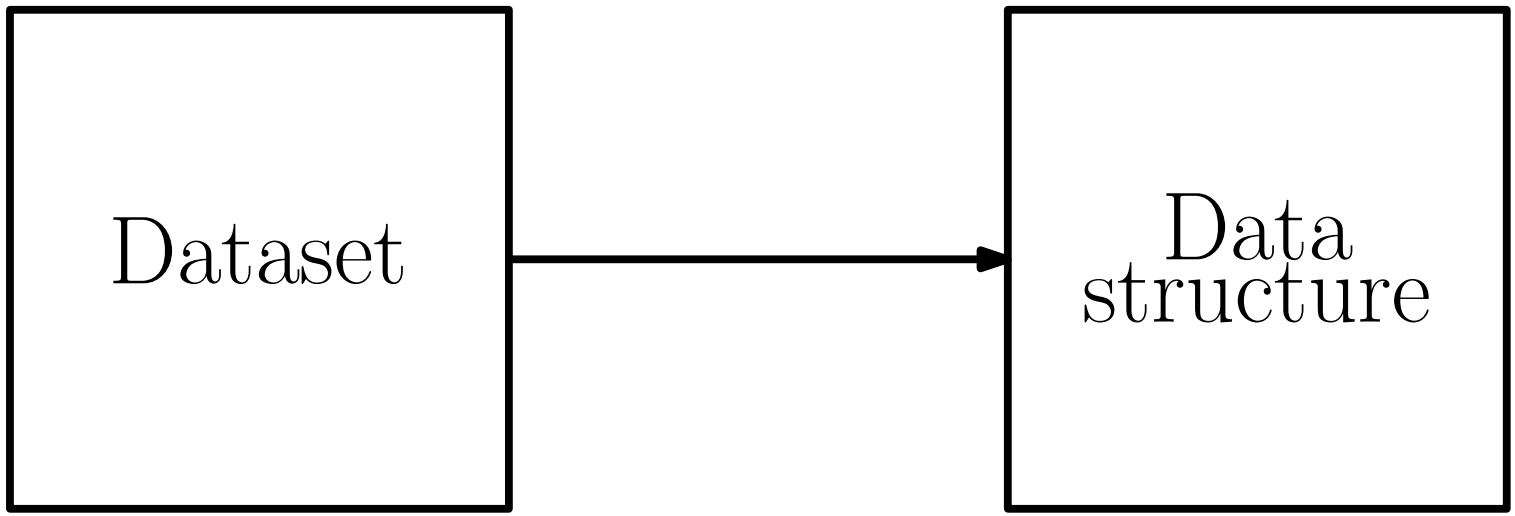

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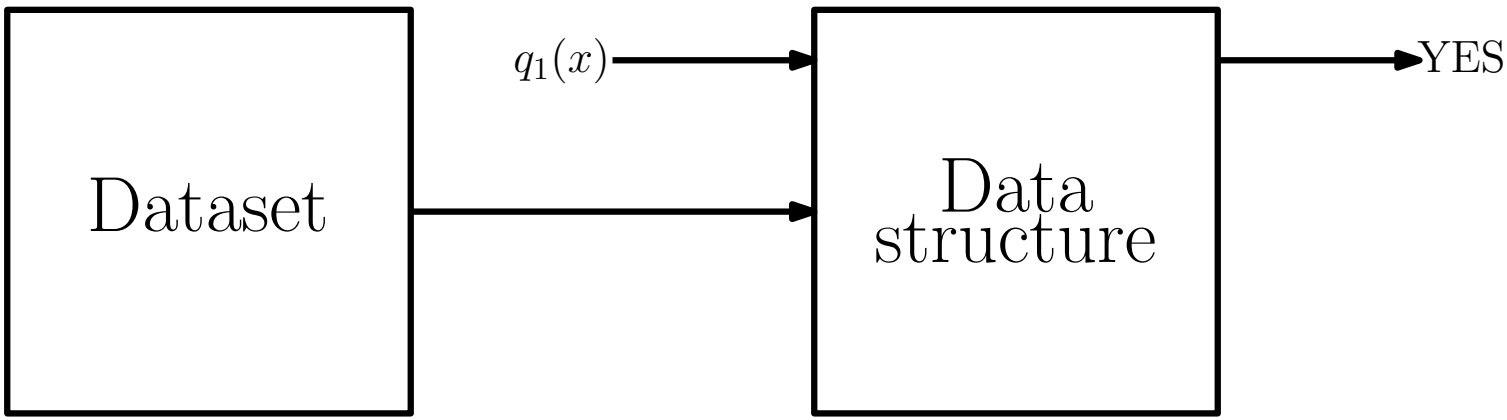

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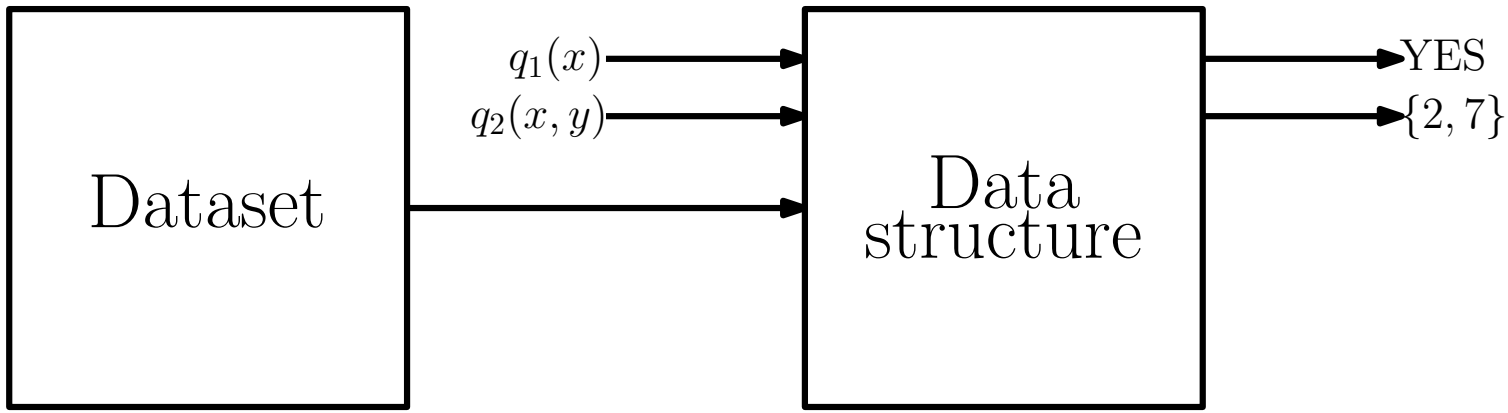

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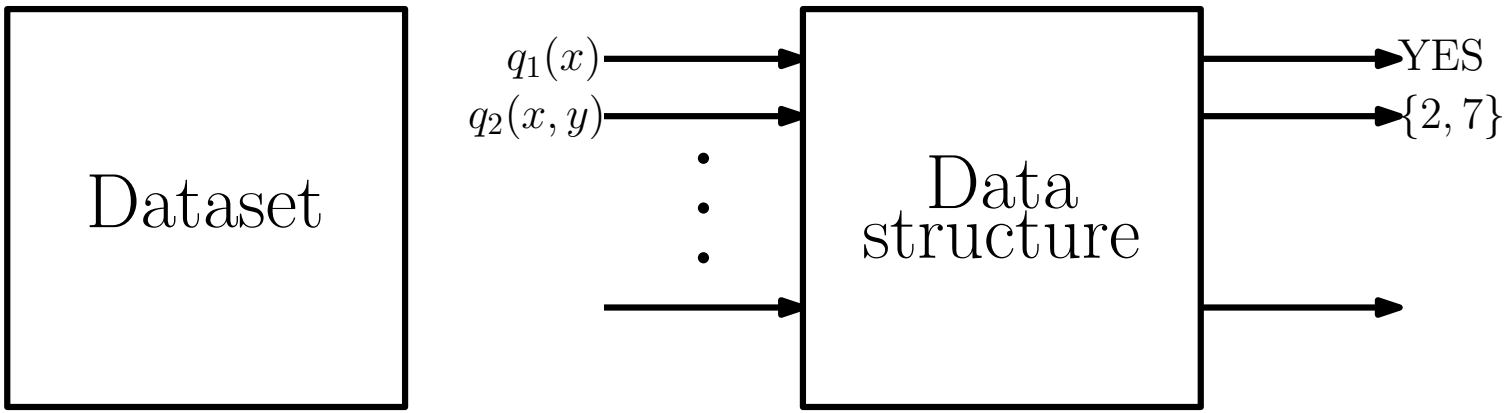

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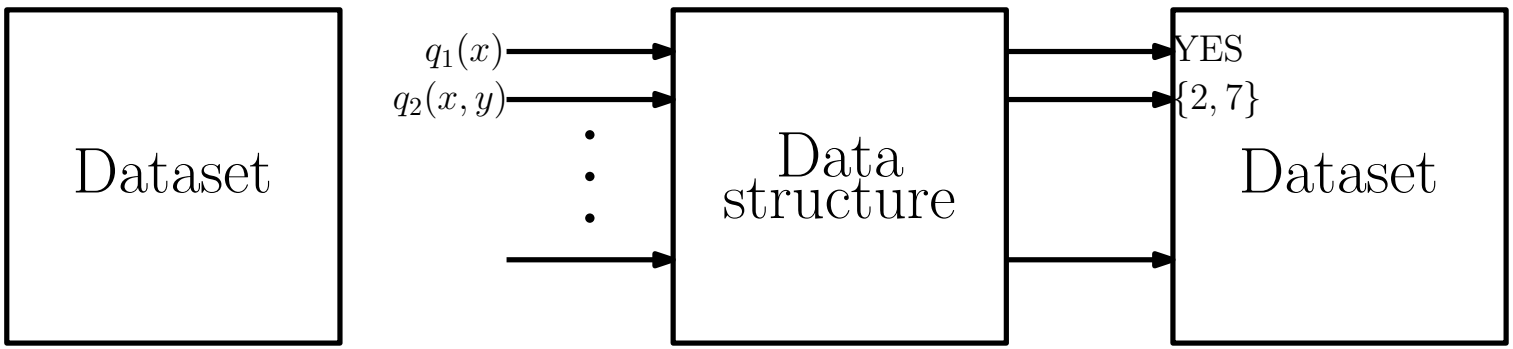

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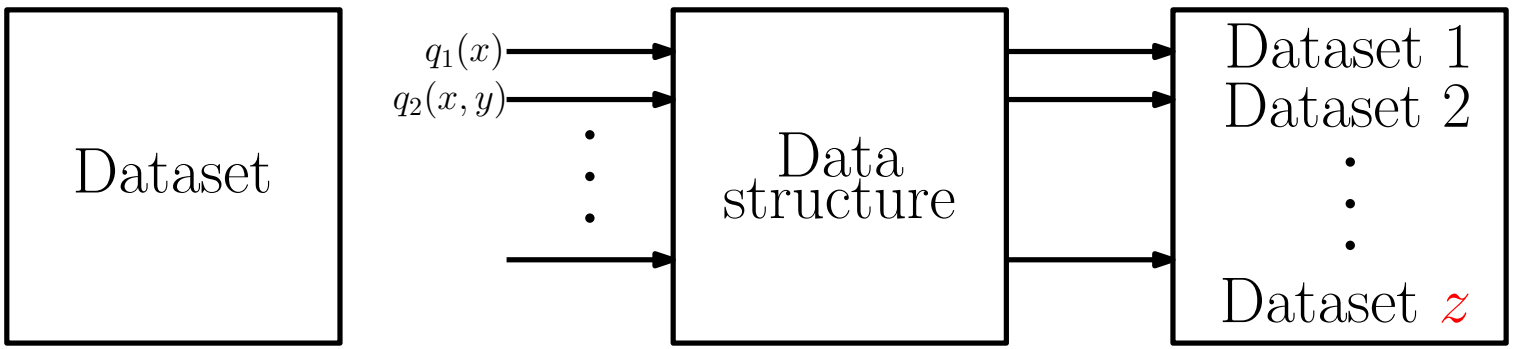

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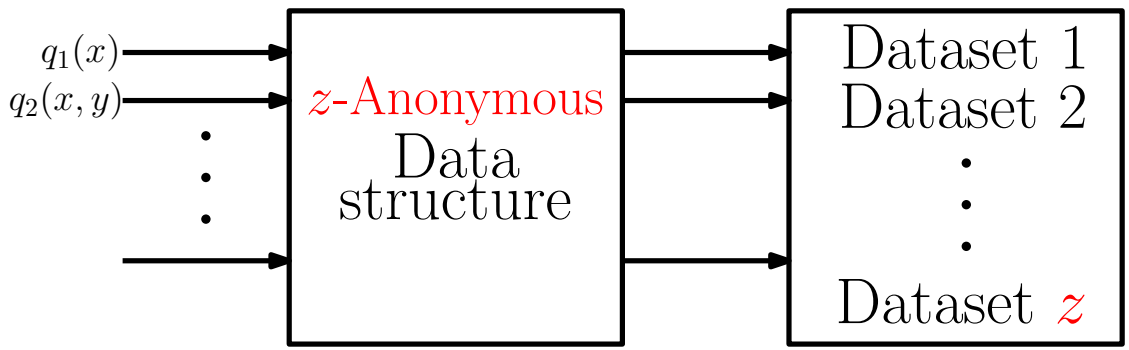

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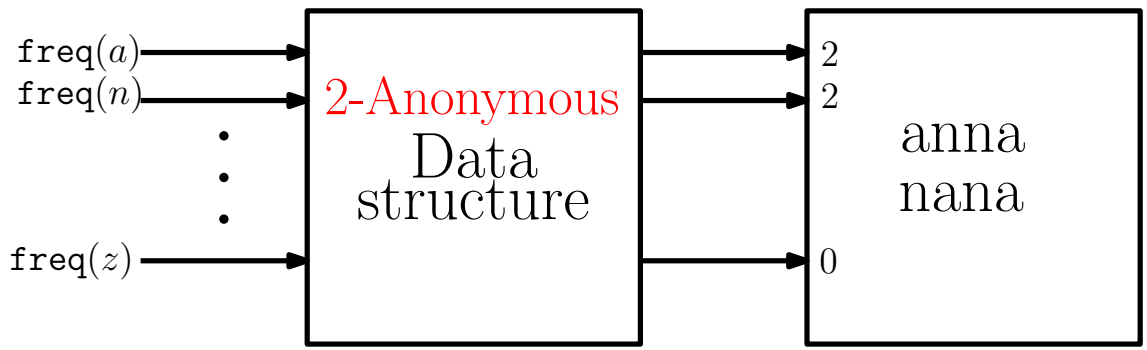

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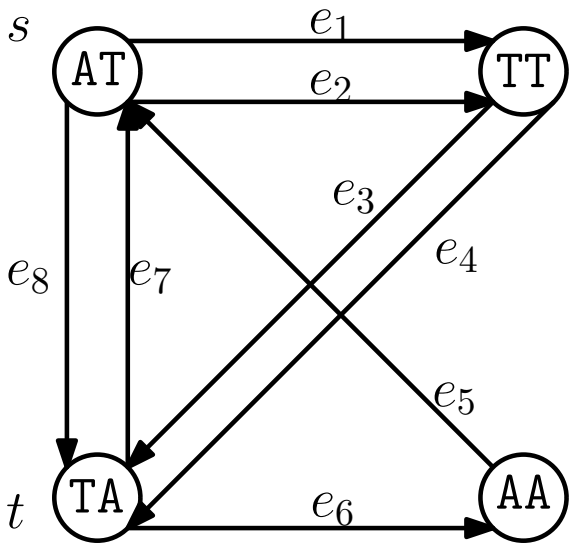

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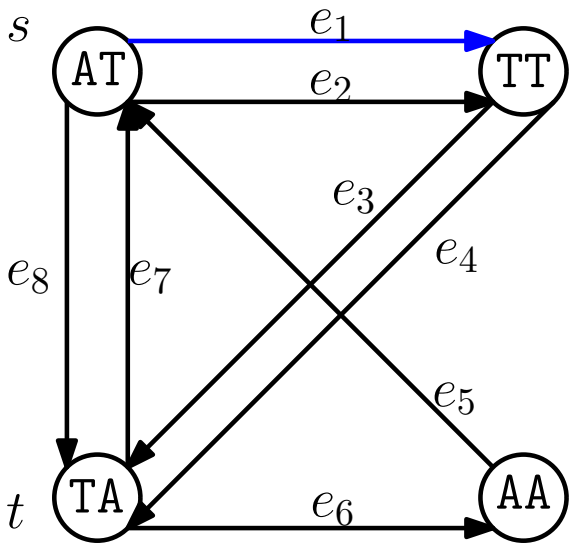


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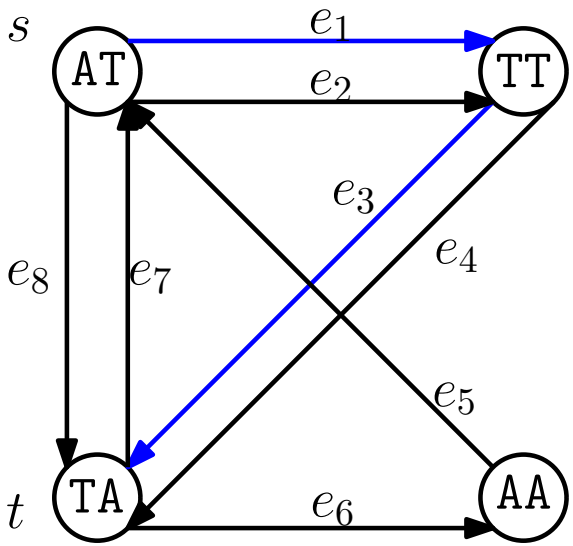


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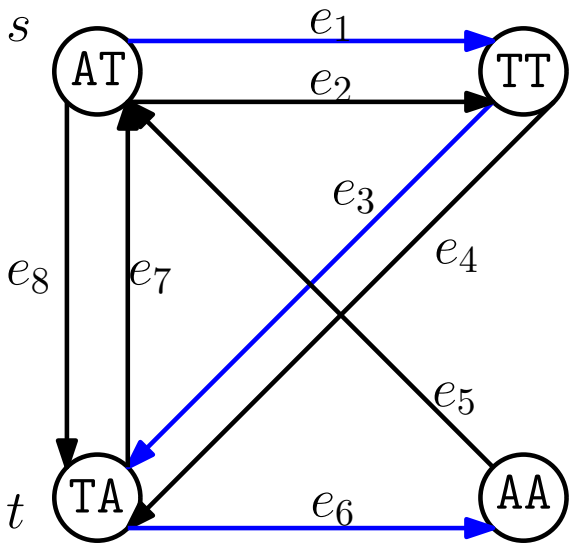


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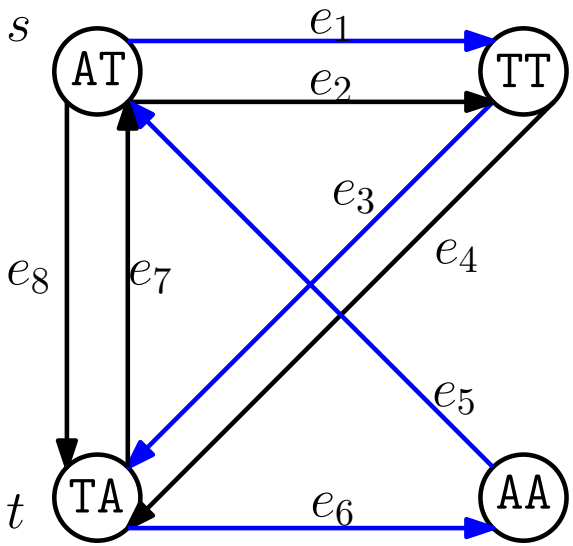


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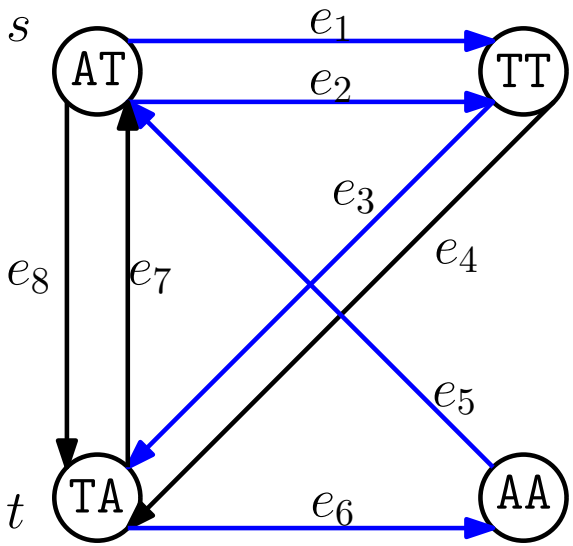


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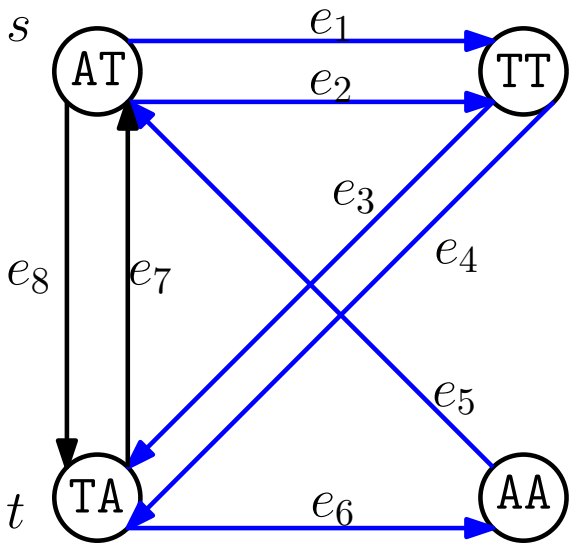


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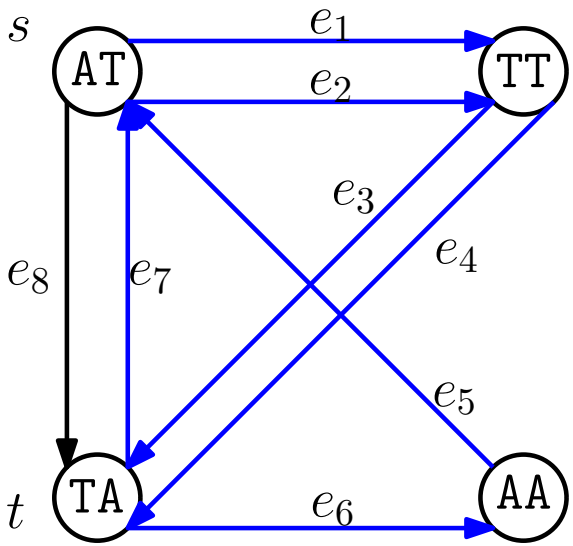


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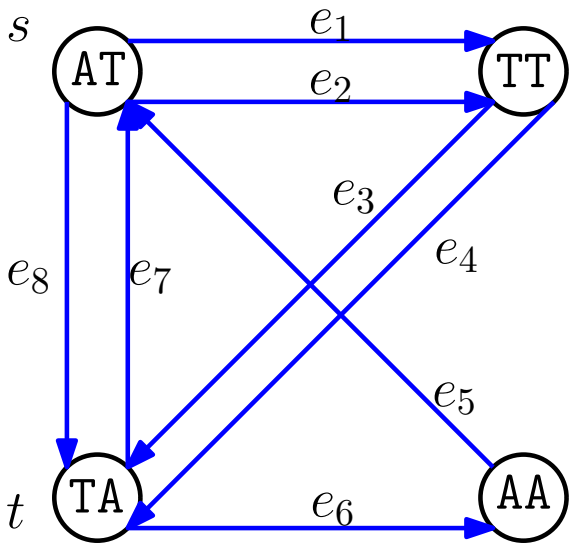


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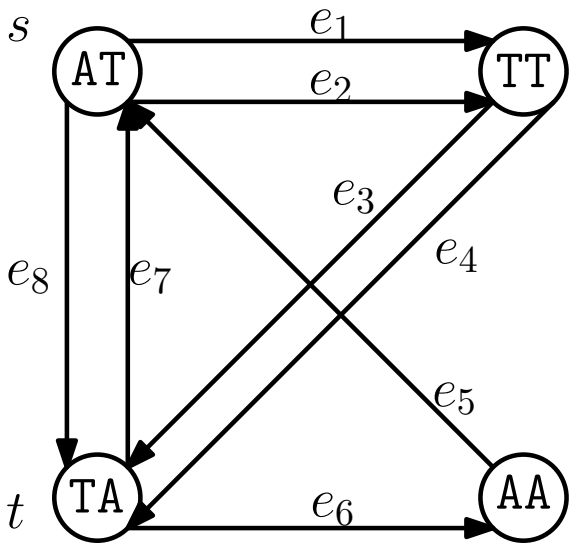


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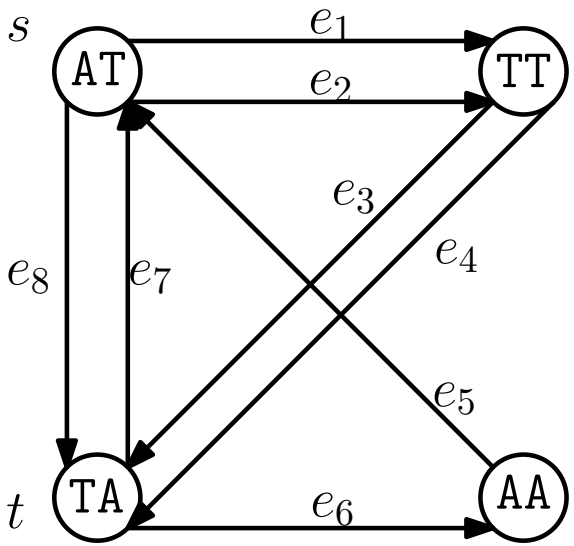


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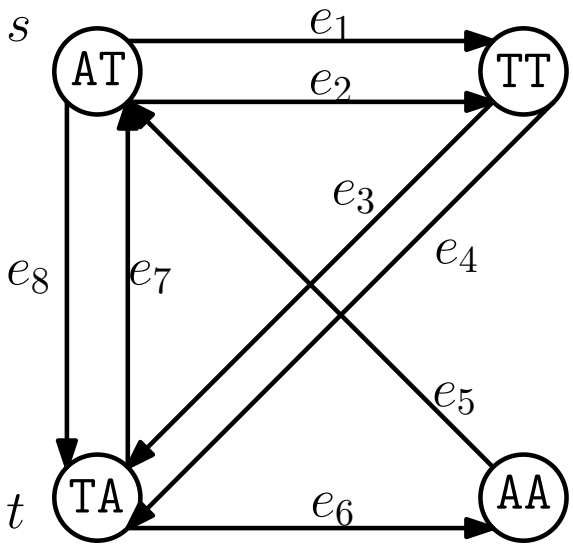


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# Reference

G. Bernardini, H. Chen, G. Fici, G. Loukides, S. P. Pissis:
Reverse-Safe Text Indexing.
ACM J. Exp. Algorithmics 26: 1.10:1-1.10:26 (2021)

# Outline

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[5] Bernardini, Marchetti-Spaccamela, Pissis, Stougie, Sweering, CPM 2021

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- We might need a completely different approach!

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- For every #, we may have different $U, V$

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Can we truncate the suffix tree in a non-uniform way?

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Can we truncate the suffix tree in a non-uniform way?

- For maximizing $d$, it suffices to cut everything at depth $d$

# Open Problem 3

**Theorem**

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Can we truncate the suffix tree in a non-uniform way?

- For maximizing $d$, it suffices to cut everything at depth $d$
- Can we really maximize the number of stored answers?

# Open Problem 4

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Can we assess $G_d = (V, E)$ faster than $O(|V|^{\omega})$ or $O(|E| + z)$?

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- $O(|V|^{\omega})$ counts
- $O(|E| + z)$ enumerates

# Open Problem 4

**Theorem**

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Can we assess $G_d = (V, E)$ faster than $O(|V|^\omega)$ or $O(|E| + z)$?

- $O(|V|^\omega)$ counts
- $O(|E| + z)$ enumerates
- However, what we need is to assess: $\alpha_d(S) \geq z$?